An Experimental Investigation of Simultaneous Multi-battle Contests with Strategic Complementarities*

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August 21, 2015

Abstract

This paper reports the results of laboratory experiments that are designed to test theoretical predictions in a multi-battle contest with complementarities. The specific setting is a game of Hex where control of each region is determined by a Tullock contest and the overall winner is determined by the combination of claimed regions. We find that in a game with only a few regions, aggregate behavior across regions is largely consistent with the theoretical predictions. However, examining individual level behavior suggests that bidders are not behaving in accordance with the model, but rather pursuing focused attacks. This intuitive behavioral approach is also found to occur in larger games where the theory is undeveloped.

JEL Classification: C7, C9, D7

Keywords: Multibattle Contests, Complementarities, Hex Game, Experiments

*The authors would like to thank the Walton College of Business at the University of Arkansas for research support. We would also like to thank seminar participants at the Southern Economics Association meetings, the University of Lyon-St. Etienne, Queen's University Belfast, the Indian School of Business, Université Bordeaux, Texas Christian University, Simon Fraser University, Chapman University, and Georgia State University. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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1 Introduction

Contests have long been used to model competitive situations like lobbying (Krueger (1974), Tullock (1980), and Synder (1989)) and patent races (Fudenberg et al. (1983), Harris and Vickers (1985, 1987)). While much of the work on contests has focused on the outcome of a single event, many situations can be better viewed as a series of inter-related contests. Changing policies and regulations often requires gaining the support of multiple lawmakers. Bringing a new product to market may require a large number of innovations. For example, Apple holds over 1300 patents related to the iPhone (Thomson Reuters (2012)). While not all of those patents may be necessary, clearly a smartphone is of no use without a user interface, a power system and an antenna. Having three user interfaces and no power system or antenna would not result in a successful product. In both of these multi-battle contest examples, the value of winning a particular contest (lawmaker or patent) depends on the combination of other contests one wins. For example, after gaining the support of a majority of lawmakers, the marginal value of an additional vote may drop to zero.

Concern about multi-battle contests goes back to discussion of the Colonel Blotto game (Borel (1921), Borel and Ville (1938), Gross (1950) Gross and Wagner (1950), and Freidman (1958)) in which two militaries allocate soldiers to a series of n battles, and in the standard version of the game, the winning side is the one that wins the most battles, similar to the lobbying example above. In these games the outcome of a given battle depends on who has the larger army in that battle and the battlefields are linked by a budget constraint which captures the fact that the number of soldiers available is fixed.

More recent work allows for asymmetric budgets and a positive opportunity cost of the resource in games with both continuous and discrete strategy spaces (see Hart (2008), Kvasov (2007), Laslier (2002), Laslier and Picard (2002), Roberson (2006), and Weinstein (2005)). Protecting pipelines and computer networks, which are only as strong as the weakest link, are examples of multi-battle contests where the parties have different objectives. In these settings one party has to win all of the battles to come out ahead while the other party only needs to win a single battle (see Clark and Konrad (2007), and Golman and Page (2009) for simultaneous move games, and Deck and Sheremeta (2012) for a sequential game of siege). Szentes and Rosenthal (2003a, b) examine a more generalized game in which one needs to win m battles to claim the prize (the majority rule is a special case where $m = \frac{N+1}{2}$), while Klumpp and Polborn (2006) allow for diminishing returns in a stochastic game of Colonel Blotto. More thorough surveys of the theoretical work in this area are available in Kovenock and Roberson (2010a) and Konrad (2009). Recent experimental work in this area includes Arad and Rubinstein (2012), who study deterministic games of Colonel Blotto, Chowdhury et al. (2013) who study both deterministic and probabilistic versions of the Colonel Blotto game, Kovenock et al. (2010) who study weakest-link games, Hortala-Vallve and Llorentr-Saguer (2010) who allow for heterogeneity of player values, and Duffy and Matros (2013) who allow for different battlefields having differing values (see Dechenaux et al. (2012) for a more general review of contest experiments).

Thus far, work on multi-battle contests has focused on situations where the overall outcome depends on simple counts of individual battle outcomes, but not the identity of the specific battles. The exception is Kovenock et al. (2015), who consider a multi-battle contest where each battle is for a given region in a 2×2 game of Hex. Hex is a simple game independently invented by Piet Hein and John Nash where each player attempts to create a path connecting opposing sides (see Figure 1). In the traditional version of Hex, individual cells in the grid are claimed alternatingly by the two players, but Kovenock et al. (2015) consider the situation where each region is allocated based upon a standard Tullock contest function.

The game of Hex can be thought of as mimicking a communications network with redundancies such as the internet, as there are multiple ways to navigate through the network and no particular relay is critical by itself. Therefore, an attack on such a network has to successfully block all possible communication routes. Hex can also be thought of as a contest in literally geographic terms like when environmentalists want to establish a greenway and developers want to build a highway. In equilibrium, contestants should compete for every region, but should compete more fiercely for certain regions.

The design and structure of Hex is such that if every region is claimed there will be exactly one winner. Hence, Hex creates an environment where there are complementarities between certain battles. Of course, there are an unlimited number of other multi-battle contests with complementarities that one could construct. Even with only four battles, one could arbitrarily assign many different sets of combinatorial values. This is the typical approach that is used in the combinatorial auction experiment literature (e.g. Banks et al. (2003), Porter et al. (2003), and Kagel and Levin (2005)). The advantage of Hex is that it creates a natural and intuitive set of combinatorial values that is easy for contestants to understand and internalize, which is important for understanding how people view complementarities. This also makes it ideal for testing theoretical results regarding complementarities in the lab.

In this paper we report the results of controlled laboratory experiments designed specifically to test the theoretical predictions of Kovenock et al. (2015) for the 2×2 game of Hex and to explore behavior in contests with complementarities more generally. We do this by comparing experimental environments with and without complementarities that have otherwise similar equilibrium predictions. We also report the results of experiments involving a larger 4×4 game board. While the size of this larger game is still relatively small, the computations involved make developing theoretical predictions for it overly cumbersome when there are complementarities, which is why Kovenock et al. (2015) restrict their attention to the 2×2 game. Such scalability is a clear advantage of a behavioral approach to studying contests. As a prelude to the results, we find that aggregate behavior in the 2×2 game with complementarities is generally consistent with the model in the sense that relative bids among different regions are in line with the equilibrium predictions. This pattern is also observed in the experiments without complementarities. However, in both conditions we observe substantial overbidding, a robust phenomenon in contest experiments (see Dechenaux et al. (2012)). Despite the aggregate success we find that players bid on winning combinations instead of bidding on every region as predicted when there are complementarities while in the absence of complementarities players do bid on every region as predicted. This same behavioral pattern is observed in the larger game as well, with the result being that on average a larger amount is bid on those cells that have a greater number of winning paths running through them when there are complementarities.

2 Model and Hypotheses

The model presented here is from Kovenock et al. (2015). Consider the 2×2 game of Hex shown in Figure 1. There are two players, X and Y. Player X controls the top left and bottom right corners of the board, while Y controls the other two corners. The objective for each player is to form a contiguous path connecting their pair of corner regions. Thus player X wins if he captures any of the following sets of regions {North, South, East, West}, {North, South, East}, {North, South, West}, {North, East, West}, {South, East, West}, {North, South}, {North, East}, {South, West}. There are also 8 winning combinations for Player Y, some of which are the same as those for player X and some that are different. For example, either player wins by capturing the set {North, South}, while only player X wins with the set {North, East} and only player Y wins with the set {South, East}. Let X* and Y* denote the set of winning sets for players X and Y respectively.

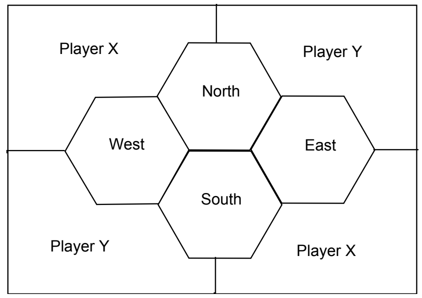


Figure 1: The 2 × 2 Hex Game of Kovenock et al. (2015)

The players are assumed to be risk neutral, have a common value for winning the game, and do not face a budget constraint. Further, the winner of each region is determined by a standard Tullock contest success function. Letting $X_r(Y_r)$ denote the investment by player X (Y) in region $r \in R = \{North, South, East, West\}$, the probability that player X wins region r is $\frac{X_r}{X_r+Y_r}$. Winning a region allows the player to use that region is completing a path between their assigned corners. Hence the probability that X wins the overall prize is given by $P = \sum_{\alpha \in X^*} \left(\prod_{r \in \alpha} \left(\frac{X_r}{X_r+Y_r} \right) \prod_{r \notin \alpha} \left(\frac{X_r}{X_r+Y_r} \right) \right)$ with a similar calculation for the probability that Y wins the prize. As any investment is forgone regardless of the outcome, player X's profit function is given by $\pi_X = VP - \sum_{r \in R} X_r$ and similarly for player Y.

The unique Nash equilibrium of this game is $X_{North} = X_{South} = Y_{North} = Y_{South} = \frac{V}{8}$ and $X_{East} = X_{West} = Y_{East} = Y_{West} = \frac{V}{16}$. Notice that each payer should invest a positive amount in each region, but should invest more in North and South which appear in more winning combinations and are thus of higher strategic value. The equilibrium calculation is straightforward, but tedious, involving the simultaneous solution to four first order conditions of the profit maximization problem for each player. In equilibrium, each player has a 50% chance of winning and an expected payoff of $\frac{V}{8}$. Notice also that aggregate investment by the two players together is $4(\frac{V}{8}) + 4(\frac{V}{16}) = \frac{3V}{4}$.[1]

For comparison, if there were no complementarities and each of the regions was valued at V_r then the standard result would hold for each region. Specifically, each bidder should bid $\frac{V_r}{4}$ for each region, resulting in a 50% chance of claiming any region and an expected profit of $\frac{V_r}{4}$ in each region. The total investment over R would be $\sum_r \frac{V_r}{2}$. If $\sum_r V_r = V$ so that the total prize was the same in the two games then without complementarities each player would invest a total of $\frac{V}{4}$, the expected profit per player would be $\frac{V}{4}$, and the total investment would be $\frac{V}{2}$. Therefore, for

[1] Details of how this equilibrium is obtained, including the ruling out of corner solutions, can be found in the appendix of Kovenock et al. (2015). Though this does not rule out the possibility of mixed strategy equilibria, any equilibrium must contain positive bids on all four regions.

the same total prize amount under complementarities total investment is higher, and the expected profits are lower.

The equilibrium investments serve as the basic hypotheses to be tested in the lab. However, given the robust result from previous experiments that people tend to overbid in simple contests we also investigate the following relative hypotheses that focus on the role of complementarities.

1. Without complementarities, bids in a region are proportional to the value of that region and thus also proportional to the equilibrium bid for the region.

2. With complementarities, bids in a region are proportional to the equilibrium bid for the region.

Before continuing to the experimental design, we briefly point out a few additional facts. First, the theoretical problem described above can be extended to an $n \times n$ size game of Hex. The equilibrium condition is determined by the simultaneous solution of $2n^2$ first order conditions. Further, the profit function itself depends on the elements of X* and Y* which each contain 2^{n^2-1} entries. So for a 4×4 game there are 32768 winning combinations for player X and 32768 winning combinations for player Y and the equilibrium level of investment depends on 32 simultaneous equations based on those winning combinations. Interestingly, while this problem quickly becomes intractable, due to the large number of wining combinations, it is trivial for a person to look at a 4×4 game and see who has won.

Finally, notice that in the 2×2 game every winning set for X is either (North, South), (North, East), (South, West), or some superset of one of these. With this in mind we define the notion of a *minimal winning set*. Minimal winning sets are the sets of cells which are sufficient for victory, but no proper subset of which is a winning set. There are also three minimal winning sets for player Y and again each involves 2 regions. As the game becomes larger, $n > 2$, it is no longer the case that every minimal winning set is of size n, although each player does always have $\sum_{i=1}^{n} 2^{i-1}$ minimal wining sets of size n.

3 Experimental Design

To explore how people behave in contests with complementarities and to evaluate the theoretical predictions of the model discussed in the previous section, we conducted a controlled experiment. The experiment involves two conditions: a treatment that involves complementarities and a control that is otherwise similar but does not involve complementarities. The two conditions were implemented using a between-subjects design.

Throughout the experiment, monetary amounts were denoted to participates in Experimental Currency (EC), which were converted to \$US at the rate of \$EC 25 = \$US 1. This exchange rate was explained to the subjects when the experiment began. Unless otherwise noted, all monetary amounts below are in EC. Subjects also received a \$US 5 payment for showing up to the lab on time for the one hour session, as is standard policy in the Behavioral Business Research Laboratory at the University of Arkansas where the experiments were conducted. A total of 6 sessions per treatment with 6 people per session were completed. The 72 participants were undergraduate students from the lab's database of approximately 2000 volunteers. While some subjects had previously participated in other economics experiments, none had participated in any related studies.

Both treatments involved three phases, each lasting for multiple repetitions of a single game referred to as a period. Subjects read phase specific instructions just prior to the start of the phase. [2] Subjects did not know how many periods were in any phase nor did they know of the existence of future phases.

Each period consisted of acquiring regions, with the difference between phases consisting in how the regions are acquired and the number of regions in the game. After all regions have been claimed, the set of regions claimed by each player is examined. Players then receive payouts based on the regions they have claimed in the game. At the start of each game all regions are unclaimed (claims do not carry over between games) and players are randomly matched with a new, unknown opponent to minimize concerns regarding repeated play or reputation building.

We now describe each phase of the experiment separately. For each phase we provide details for the treatment with complementarities and then describe how the control was implemented to maintain parity.

Phase 1: Sequential Play in a 2 × 2 Game with No Investment

The purpose of the first phase of the experiment was two-fold. First, it allows players to discover the strategic value of each region to both players without explicitly being instructed about the relative importance of the North and South regions, which might bias bidding behavior. Second, it provides an opportunity for the subjects to earn money which can be used as an endowment during the later phases of the experiment where one of the players will necessarily lose money.

The first phase consisted of 10 games on a 2 × 2 Hex board. The regions were actually squares instead of hexagons, but the arrangement was such that the winning combinations were the same

[2] Copies of the instructions are provided in Appendix A.

as discussed in the previous section. In each game one player moved first and was able to claim a region (at no cost) by simply clicking on their screen. The second player could observe the choice of the first player and then select a region to claim. This choice was revealed to the first mover who could then claim a second region leaving the final region for the second mover. Each player alternated between being the first and second mover across the 10 games in phase 1.

For the complementarities treatment, the value of completing a winning set was 48, for reasons that will be explained later. In this case, the first mover should pick North or South initially. The second mover should then select whichever remains unclaimed of North or South in the hope that the first mover makes a subsequent mistake. The first mover should then make a wining move and hence the first player should always win. This first mover advantage holds in the sequential game of Hex regardless of the board size, In fact, the board game from Parker Brothers is played sequentially on an 11 $\times$ 11 board. Each subject started phase 1 with an endowment of 160 and would end phase 1 with a cumulative profit of 400 (= 160 + 5 x 48 + 5 x 0) if everyone played optimally.

In the no complementarities treatment, the North and South regions were valued at 8 while the other two regions was valued at 4 each, for reasons that are explained later. Thus, the first mover should take either North or South and the second mover should take the other of these relatively high valued regions. That is, the sequence of actions should be the same in the control as in the complementarities treatment. Each subject started phase 1 with an endowment of 280 and would end phase 1 with a cumulative profit of 400 ($= 280 + 10 \times (8 + 4)$)) if everyone played optimally. Thus, subjects in both conditions should start phase 2 with the same earnings.

Phase 2: 2 $\times$ 2 Game with Regions Decided by Contests

Phase 2 constitutes the primary portion of the experiment. In this phase, subjects played 20 games on a 2 $\times$ 2 board, where the allocation of each region was independently determined by a Tullock contest. In this phase the players privately and simultaneously submitted bids for each region, as shown in Figure 2. For the treatment with complementarities, the prize for winning was again 48. As shown in Table 1, the equilibrium prediction is that a subject will bid 6 on North and South and 3 on East and West for a total bid of 18. Because each subject has a 50% chance of receiving the prize in equilibrium, the expected profit to a player is 6.

For the control, North and South were valued at 8 and East and West were valued at 4, see Figure 3. As shown in Table 1, the equilibrium bid is 2 in North and South and 1 in East and West for a total bid of 6. Because a player wins each contest with a 50% chance in equilibrium, the expected profit is 6, just as in the complementarities treatment. That is, the value of the regions in the control condition are set such that the incentive level is the same for both treatments. Further, the values in the control condition are set such that the relative value of each region is held constant across treatments. A benefit of this is that it affords a clearer test of how the complementarity impacts bidding behavior as well as allowing for a test of the standard contest model as the value of the prize changes.

	No Complementarities	With Complementarities
Value of Winning	North, South worth 8 East, West worth 4	Completed Set worth 48
Equilibrium Bid on North, South	2	6
Equilibrium Bid on East, West	1	3
Expected Profit per Player	6	6

Table 1: Summary of Experiment Parameters and Predictions for 2 × 2 Games

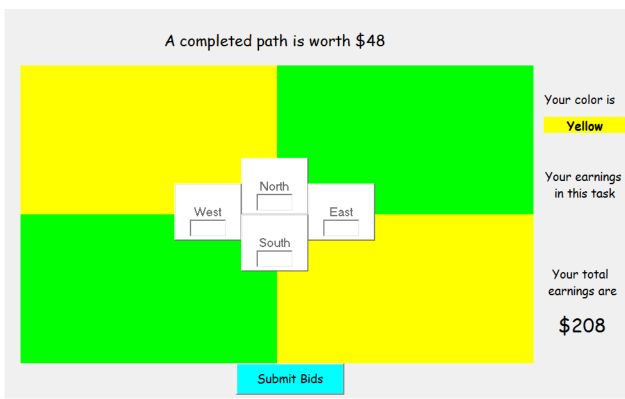


Figure 2: Screenshot of 2 × 2 game with complementarities

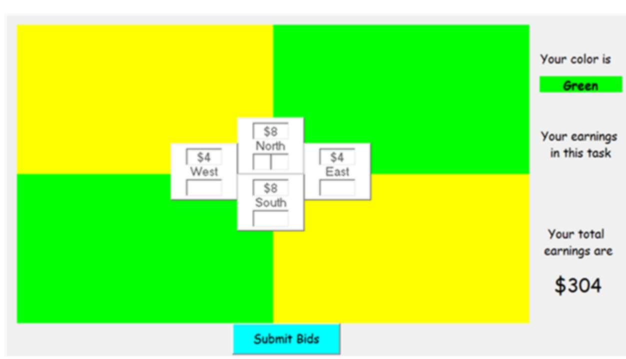


Figure 3: Screenshot of 2 × 2 game with no complementarities

Figure 4: Screenshot of 4 × 4 game with complementarities

Phase 3: 4 × 4 Game with Regions Decided by Contests

This phase provided an opportunity to explore behavior in a more complicated environment. The procedures for this phase were identical to those in phase 2, except that the game board was increased to 4 × 4 and subjects only played the game 5 times (and thus placed the same total number of bids as in phase 2 since there are four times as many regions in phase 3). Note that this game is analytically unwieldy due to the large number of first order conditions. Figure 4 shows a sample outcome of the 4 × 4 game with complementarities. The value of a winning set remained 48 for the treatment with complementarities. For the control treatment, each of the 16 regions was valued at 3 so that the total value was the same between the two treatments, allowing for an evaluation of the impact of incentives and additional variation in prize values for standard stand-alone contests. Given the exploratory nature of this game and the inherent increase in complexity when there are complementarities, this phase was always conducted last so as to (1) not influence behavior in the 2 × 2 game which is the main focus of the experiment and (2) provide subjects an opportunity to learn in the simpler game so that observed behavior in this game is meaningful. While these two concerns are not as important when there are no complementarities, the two treatments are kept parallel to make direct between treatment comparisons.

4 Behavioral Results

We present the results for each phase of the experiment in separate subsections.

4.1 Behavior in Phase 1: Sequential Play in a 2 $\times$ 2 Game with No Investment

We begin our analysis with behavior in phase 1 of the experiment. Overall, players made an optimal choice 98% of the time in the no complementarities case, where optimal is defined as claiming the higher valued region if it is available. For the more complicated sequential Hex game, optimal behavior, defined as following the subgame perfect equilibrium strategy conditional on the decision point, is observed 95% of the time. Further, no suboptimal behavior was observed in the last three rounds for either treatment. This finding clearly indicates that subjects understood the strategic value of the different regions before beginning phase 2.

4.2 Behavior in Phase 2: 2 $\times$ 2 with Regions Decided by Contests

We now turn to analyzing behavior in phase 2. Given the symmetry in the game, regions are standardized around a vertical line drawn through the center of the game and reported with respect to the position of a region from the perspective of the Yellow player who is trying to complete a path from the top left to the bottom right. For the 2 $\times$ 2 game this means that for reporting purposes bids Green placed nominally for the West region are combined with Yellows' bids for the East and vice versa. The average bid for each region is shown in Table 2.

We begin by considering the no complementarities treatment. The left column of Table 2 summarizes bid behavior in this case. Several interesting features of the data are readily apparent. First, for all four regions players are bidding almost twice the equilibrium value on average. This is clear evidence that the subjects are not bidding according to the equilibrium predictions, but this pattern of overbidding is common as discussed in Dechenaux et al. (2012). The reasons for such overbidding are explored in Cooper and Fang (2008). The second main feature of the data is that players are bidding basically the same amount for North and South, consistent with Hypothesis 1. The players are also bidding nearly identically on average for East and West, also consistent Hypothesis 1. Finally, players are bidding twice as much for North and South as for East and West, again consistent with Hypothesis 1.

Table 3 reports the estimation results from regression analysis where the dependent variable is the amount bid on a region relative to the equilibrium bid for that region and South and West are dummy variables for those respective regions and Side is a dummy variable that takes the value of 1 for the East and West regions and is 0 otherwise. Thus the omitted case is the North region. Side captures the effect of being in the East or West, low value, regions while West captures any

	No Complementarities		With Complementarities	
	Observed	Percent Overbid	Observed	Percent Overbid
North	3.81	91%	7.38	23%
South	3.75	88%	8.36	39%
East	1.89	89%	3.56	19%
West	1.85	85%	4.65	55%
North / South	1.02		0.89	
East / West	1.02		0.77	
$\frac{North+South}{East+West}$	2.03		1.92	

Table 2: Average Bids by Region in the 2 × 2 Games

differential effect between the East and West regions. Standard errors are clustered at the session level and a random effect is included for each individual bidder. The joint lack of significance for the dummy variables in Table 3 provides statistical support in favor of Hypothesis 1. To test whether or not players are bidding according to the equilibrium predictions, involves comparing the constant term in Table 3 to the predicted value of 1. This hypothesis can be rejected in favor of systematic overbidding at even the 1% significance level. This provides the basis for Finding 1.

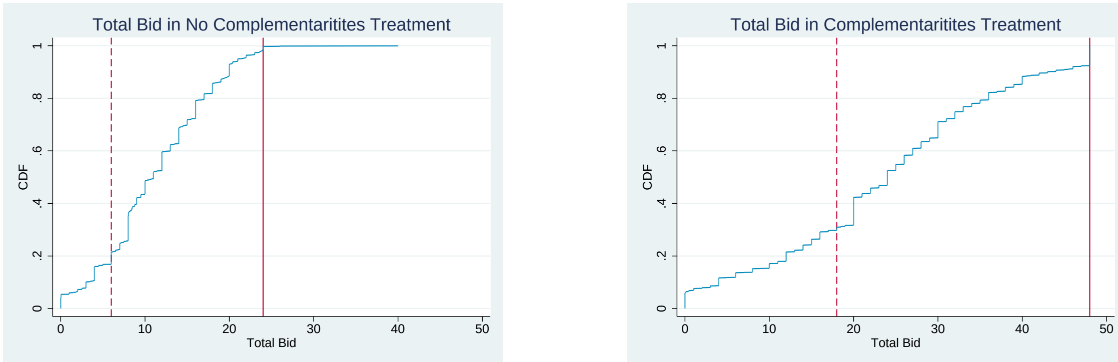
Finding 1: Without complementarities, the relative bids on two regions is equal to the relative equilibriums bids for those regions.

The overbidding in the absence of complementarities can be seen in Figure 5(a), showing the cumulative distribution of the total amount spent by players in the no complementarities treatment. The dashed line indicates the equilibrium total spending $= 6(= 2onNorth + 2onSouth + 1onEast + 1onWest)$, while the solid line marks the total prize value (24) for all four areas. Figure 5(a) suggests that total expenditure is almost uniformly distributed and centered around $\sum_r \frac{V_r}{2}$.

	Coefficient	Robust Std. Error
Constant	1.9070***	0.0447
Side	-0.0535	0.0736
West	0.0238	0.0278
South	-0.031	0.0385
***indicates 1% Confidence Level		

Table 3: Region Game Ratio of actual bids to theoretical bids, regressed on region dummies

Turning to the treatment with complementarities, the data presented in the right column of Table 2 reveal that subjects are again overbidding in each region, but not as dramatically as in the no complementarities case. Table 2 also provides strong evidence in support of Hypothesis 2. The average bids are very similar in North and South as predicted. Further, average bids are similar in



(a) No Complementarity Treatment (b) Complementarity Treatment

Figure 5: Total Spending By Complementarity Treatment

the East and West, as predicted. Finally, bids are predicted to be twice as high in the high strategic value regions (North and South) as in the low strategic value regions (East and West) and this is what is observed. Statistical evidence is provided in Table 4, which reports a similar regression to that done for the no complementarities treatment. Hypothesis 2 is supported by the joint lack of significance on the three dummy variables. This provides the basis for Finding 2.

Finding 2 With complementarities, the relative bids on two regions are equal to the relative equilibrium bids for those regions.

	Coefficient	Robust Std. Error
Constant	1.2297***	0.1503
Side	0.2409	0.2457
West	-0.2066	0.3342
South	0.1626	0.1318
***indicates 1% Confidence Level		

Table 4: Path Formation Game Ratio of actual bids to theoretical bids, regressed on region dummies

In this case, however we cannot reject the hypothesis that the constant term equals 1 at the 5% significance level. This is due to the lower overbidding that we find when complementarities are present. Figure 5(b) gives the cumulative distribution of total spending in the path formation game. The dashed line indicates the theoretical equilibrium total spending, while the solid line is where spending equals the prize. As in Figure 5(a), total bidding appears to be relatively uniform. Hence, the reduction in overbidding is driven by the fact that the predicted total bid is a greater percentage of value in this treatment.

The above analysis focuses on aggregate behavior and in general the results seem to support the relative theoretical predictions. However, these results are masking a distinct behavioral pattern. Theory predicts that a player should bid on every region, but this pattern is observed in only 35% of the realizations, see Figure 6. Only 1% of the time does the set of regions on which a subject bid constitute a non-winning set. The missing majority of observations, 64%, are where subjects bid on a path through the game such as {North, West}, or a set of three regions containing a winning set such as {North, South, West}.

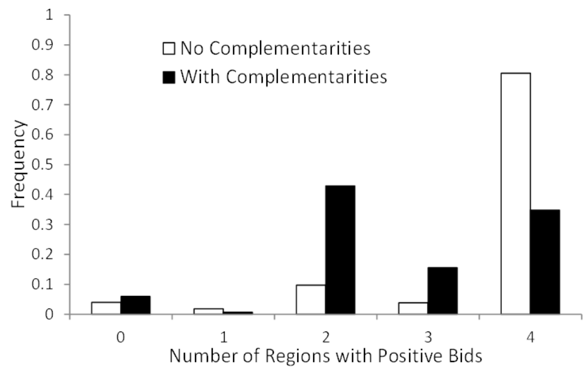


Figure 6: Number of Positive Bids in 2 × 2 Game

Figure 7 shows how subjects allocate their bids among the four regions. Each triangle plot omits the fraction of the total bid that was placed on South. The fraction bid on South is reported at the top of each plot and increases in successive plots. Hence, the size of the triangles shrink as a lower fraction of the total bid is allocated to the three remaining regions. Horizontal lines from the upper left edge of the triangle give the fraction of the total bid that was placed on North. The fraction bid on East is given by the scale on the upper right edge of the triangle with positively sloped gridlines and the fraction bid on West is found on the bottom edge with the negatively sloped gridlines. The size of each marker indicates the frequency of a particular bidding strategy.

From Figure 7, it is clear that subjects frequently bid on a minimum winning set, splitting their bids nearly equally over the regions in that set and bidding little if any on the other regions. For example, in the large triangle plot in Figure 7, where the percent bid on South is close to zero, there is a cluster of observations where essentially half of the total amount bid is placed on North and half on East and nothing is bid on West. Bids focusing on the North-South set are found at the top of the 6th plot and bids focusing on the West-South set are found in the lower right of the 6th plot. There are some instances where a subject bid uniformly across the four regions as seen in the middle of the 4th plot. The 4th plot also reveal a lack of behavior consistent with the

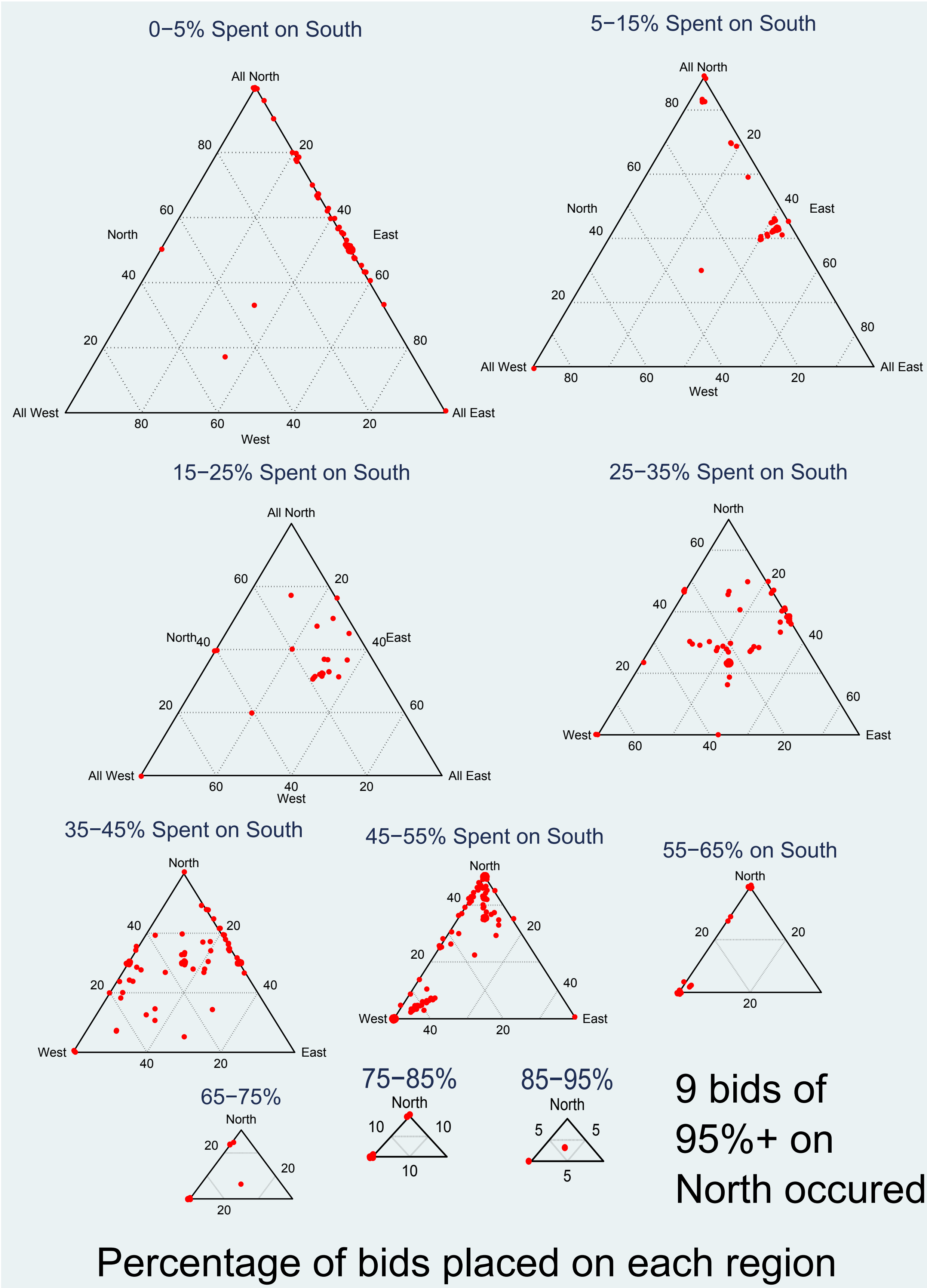


Figure 7: Fraction of Total Bid Placed On Each Cell

equilibrium and behavioral prediction.

It is worth noting that if one randomly selects a minimum winning set and bids uniformly on it, the result would also be average bids that are twice as high for North and South as for East and West. Further, if the total bid equaled half of the value of winning, as occurred in the independent contests with the no complementarities treatment, then one would bid 12 on North and South with two-thirds probability and would bid 12 on East and West with one-third probability. The result would be an average bid of 8 for North and South and an average bid of 4 for East and West, exactly the pattern revealed in Table 2. Hence it appears that theoretical predictions of Kovenock et al. (2015) are valid in the aggregate, but not for the right reason.

Given that subjects often restrict themselves to bidding on a minimal winning set, it is interesting to consider what the equilibrium would be if such a restriction was binding.

Proposition: If players are restricted to bidding on minimal winning sets then player X will bid on {North, South}, {North, East}, and {West, South} all with equal probability and will bid $\frac{V}{8}$ for both regions in the selected set. Similarly, player Y will bid on {North, South}, {North, West}, and {East, South} all with equal probability and will bid $\frac{V}{8}$ for both regions in the selected set.

Proof: See Appendix B.

This restriction essentially removes the complementarities present in the unrestricted game. For the parameters used in phase 2 of the experiment, the equilibrium of the restricted game would involve pacing a bid of 6 on each region of some the randomly selected minimal winning set. Thus, the average bid on North and South would be 4 since each is present in two-thirds of the winning sets and the average bid for East and West would be 2 since each appears in one third of the minimal winning sets. These amounts are approximately half the averages shown in Table 2. Thus it appears that subjects are treating the unrestricted game as a restricted game, but systematically overbidding. Rather than bidding a total of $\frac{V}{4}(=\frac{V}{8}+\frac{V}{8})$, they are bidding a total of $\frac{V}{2}(=\frac{V}{4}+\frac{V}{4})$, similar to the level of overbidding observed in contests without complementarities.

Of course, self-imposing the restriction to bid on minimal winning sets is not optimal. Table 5 compares the average realized profits from pursuing different strategies. Path bidding is not profitable and generates significantly lower profits than bidding on all four regions at the 95% significance level. Further, path bidding does not generate significantly different profits than bidding on non-winning sets.

The above analysis coupled with Figure 6 might suggest that there may be two types of subjects: those that bid on minimal winning sets and those that bid on every region. However, this is not the case. In fact, over 80% of the subjects in the treatment with complementarities follow both strategies at some point during phase 2. Regression analysis, reported in Table 6, indicates that subjects do not change the frequency with which they bid on every region over the course of experiment, nor do they change their strategy based on the behavior of their opponent in the previous round. However, they are more likely to switch their strategy, either from bidding on a minimum winning set to bidding on every region or vice versa, after incurring a loss.

One can construct a strategy transition probability matrix, T, for players in the complementarity treatment comparing the strategy a player used in round n and the strategy the player used in round $n+1$. This transition matrix is shown in Table 7. T^N would identify the frequency of strategies after N iterations. Taking the limit as N goes to infinity would not converge to a single state, instead the convergence goes to None at .066, One at .447, Two at .138, and Three at .349 frequencies. Thus this is not a simple case of players converging towards a single strategy, and so alternative explanations should be considered.

	Mean	Standard Error
All Four Regions	2.683	1.520
Path	-1.227	1.212
Non-Winning Set	-2.200	1.670

Table 5: Mean Profits by Positive Bids in Complementarity Treatment

	Coefficient	Standard Error
Previous Round Profit	-0.0022989***	0.0005169
Opponent Bid on Path in Previous Round	0.0318251	0.0346762
Period	-0.0016532	0.0028953
Constant	0.1695071***	0.0408861
∗∗∗indicates 1% Confidence Level		

Table 6: Switching Between Strategies in Complementarity Treatment

	None	One	Two	Three
None	.564	.256	.026	.154
One	.035	.812	.073	.080
Two	.019	.221	.634	.125
Three	.030	.105	.046	.819

Table 7: Strategy Transition Probabilities in Complementarity Treatment

4.3 Behavior in Phase 3: 4 $\times$ 4 Game with Regions Decided by Contests

We now turn to behavior in the exploratory third phase of the experiment. Figure 8 shows the average bid in each region for the no complementarities treatment. In this case, each region had a value of 3 and thus the equilibrium bid is 0.75. The average observed bid across all 16 independent regions was 1.09 and there are no major differences between regions. The average rate of overbidding was 45%, which is smaller than in phase 2. One reason for this drop in overbidding is that subjects only bid on every region 64% of the time with the other observations typically involving a few ignored regions. Once non-bids are accounted for the rate of overbidding is similar to that in phase 2, suggesting that the level of incentives are not driving overbidding directly. Of course, failure to bid on every region could be due to fatigue or the large number of choices that had to be made, which may in turn be influenced by the level of the stakes.

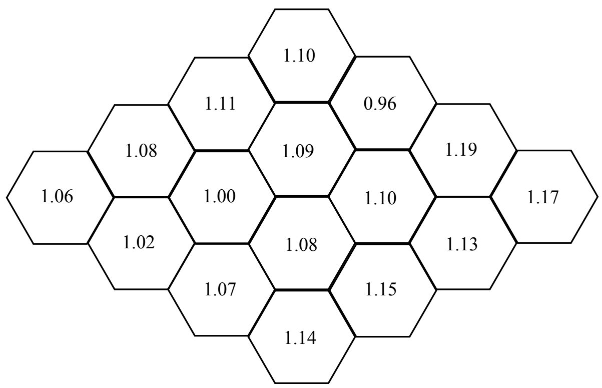
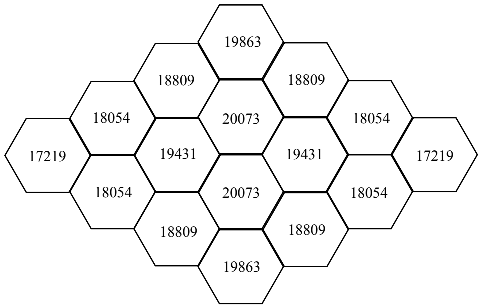


Figure 8: Average Bid By Area

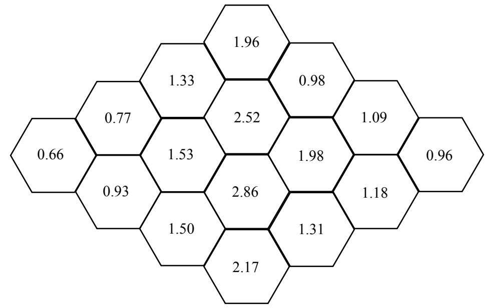
In the 4 $\times$ 4 game with complementarities, the observed bid for each region is related to the number of winning sets that contain the region. Figure 9(a) shows the number of winning sets and Figure 9(b) the average observed bid by region for this case. The correlation between the two measures is quite high at $\rho = 0.898$. However, this masks individual heterogeneity that is distinctly out of equilibrium. Rather that bidding on every region as predicted, 36% of bids are for a 4 region set and only 31% of bids involve more than half of the regions, see Figure 10.[3] This figure also reveals a non-trivial number of subjects bidding on 5 to 8 regions, giving themselves multiple possible winning sets. The same phenomenon is occurring in the 2 $\times$ 2 game when people in this

[3]Note that 3% of bids are for only one, two or three regions and thus could not result in a winning set. In this treatment 4% of the bids are zero for all 16 regions. For comparison, 3% of bids in the no complementarities treatment were zero in all 16 regions.

treatment bid on 3 regions as shown in Figures 7 and 8.



(a) Number of Winning Sets Containing Region



(b) Average Bid By Region

Figure 9: Comparison of 4 × 4 Winning Sets and Average Bids

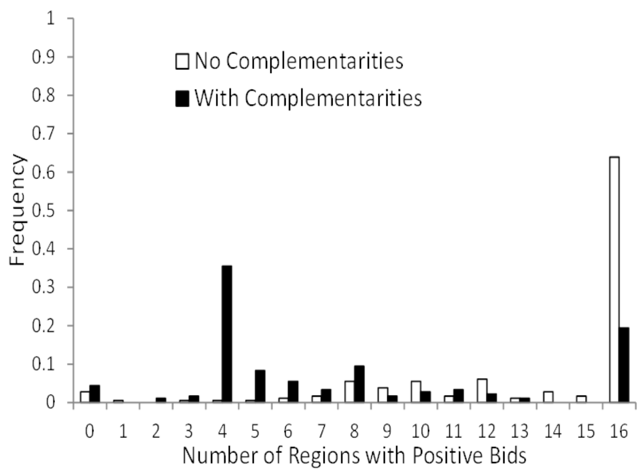


Figure 10: Number of Positive Bids in 4 × 4 Game

5 Concluding Remarks

Contests have long been used to model a wide array of activity. Recently, researchers have begun to look at ever more complicated strategic situations where outcomes are based on a series of interconnected battles. One such example is the recent model by Kovenock et al. (2015), which focuses on the game of Hex, but is representative of a wider class of games that involve complementarities in outcomes. This model is particularly relevant for the protection of networks that have built in redundancies, such as telecommunications or computer networks. Unfortunately, the tradeoff of additional complexity is often tractability. For example, Kovenock et al. (2015) restrict attention to a 2×2 game board.

In this paper we report the results of a controlled laboratory experiment designed to test the predictions of the Kovenock et al. (2015) model specifically and also explore how players actually approach more complicated contests. After all, if people really solved the problem the way theorists do, there would not be much point in researchers considering such models as the solutions would already be apparent. What we find from the experiment is that in aggregate the model's predictions accurately reflect the relative bids in the different regional battles. Overbidding is observed with and without complementarities, the later result being consistent with previous contest experiments. However, the model's predictions are correct for the wrong reason. Rather than fighting for every region in accordance with the model, subjects actually concentrate on specific winning combinations and largely ignore the other battles. As in the case of independent values, people tend to bid just less than half of the prize value, but in the Hex game they often spread that amount along a minimal winning set. There is evidence that some of the players will switch from a minimal winning set to playing all four regions in response to a loss, with losses also causing switching away from playing all four regions to a minimal winning set. It appears that some players are open to switching between strategies, while others will pick a type of strategy and continue with it.

An advantage of a behavioral approach to investigating contest behavior is that one can investigate situations beyond what is computationally attractive. Here we also considered a more complex 4×4 that involves 32768 wining combinations for each player. Again aggregate behavior is such that bids are higher on average for regions that are in more winning combinations. However, this aggregate pattern is again explained by individuals focusing on some specific winning combination. This finding indicates that overbidding (which is lower in the presence of complementarities because predicted spending is higher) and concentrated attacks are robust behavior. The implications of these patterns are potentially quite broad as researchers attempt to identify strategies for attack and defense of networks in many naturally occurring applications such as cyber-security or supply chains.

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6 Appendix A: Subject Instructions

On the following pages there are two sets of instructions. The first set is for the treatment with no complementarities and the second set is for the treatment with complementarities. Text in [brackets] was not observed by subjects.

[No Complementarities Phase 1]

This is an experiment in the economics of decision making. You will be paid in cash at the end of the experiment based upon your decisions, so it is important that you understand the directions completely. Therefore, if you have a question at any point, please raise your hand and someone will assist you. Otherwise we ask that you do not talk or communicate in any other way with anyone else. If you do, you may be asked to leave the experiment and will forfeit any payment.

The experiment will proceed in three parts. You will receive the directions for part 2 after part 1 is completed and for part 3 after part 2 is completed. What happens in part 2 does not depend on what happens in part 1, and what happens in part 3 does not depend on what happens in part 1 or 2. But whatever money you earn in part 1 will carry over to part 2 and whatever you earn in part 2 will carry over to part 3.

Each part contains a series of decision tasks that require you to make choices. For each decision task, you will be randomly matched with another participant in the lab. None of the participants will ever learn the identity of the person they are matched with on any particular round.

In each task you have the opportunity to earn Lab Dollars. At the end of the experiment your lab dollars will be converted into US dollars at the rate $25 Lab Dollars = $US 1.

You have been randomly assigned to be either a "Green" or a "Yellow" participant. You will keep this color throughout the experiment. For every task you will be randomly matched with a person who has been assigned the other color. Your color is indicated in a box on the right side of your screen. This portion of your screen also shows you your earnings on a task once it is completed and your cumulative earnings in the experiment. You will start off with an earning balance of $280.

In the center of your screen you can see a map with four regions: North, South, East, and West. The North is worth $8. The South is worth $8. The East is worth $4. The West is worth $4. This information is displayed at the top of your screen above the map. In part 1 of the experiment, when it is your turn you can claim any one unclaimed region on the map by clicking on it. When you claim a region, that part of the map will be colored in with your color and your earnings will be increases by the value of the region you claimed. When the person you are matched with claims a region, it will be colored with that person's color and that person's earnings will be increased by the value of the claimed region.

On each task Green and Yellow alternate turns until all of the regions on a map are claimed. Once all of the regions are claimed the task is complete. At that point, you will be randomly re-matched with another participant for the next task. Which color gets to go first alternates between tasks. You will go through this process several times.

[No Complementarities Phase 2]

Part 2 of this experiment is very similar to part 1. The map is the same, as are the values of each region. Your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

What is different is how the regions are claimed. Now you and the person you are matched up with have to bid for each of the four regions at the same time. However, any amount you bid on a region is deducted from your earnings regardless of whether or not you get to claim the region. Since you have to pay what you bid, the sum of your bids for the four regions cannot exceed the amount of earnings you have when placing your bids.

Bidding for a region works in the following way. The chance that you claim a region is proportional to how much you bid relative to the total amount bid for that region. For example, suppose that Yellow bid \$6 for the North and Green bid \$2 for the North then the chance that Yellow would claim North is 6/(6+2) = 6/8 = 75%. The chance that Green would claim North is 2/(6+2) = 2/8 = 25%.

As another example, suppose that Yellow bid \$0 for the North and Green bid \$0.25 for the North then the chance that Yellow would claim North is 0/(0+0.25) = 0%. The chance that Green would claim North is 0.25/(0+0.25) = 100 %.

If both bidders bid \$0 for a region then each would claim the region with a 50% chance.

You and the person you are matched with will both privately and simultaneously place your bids for all four regions at one time. The computer will then determine who claims each region based upon the probabilities associated with the bids. Each region will turn Yellow or Green to indicate who claimed that region.

As before, whoever claims a region will receive the value for that region and have that value added to their earnings. Suppose Yellow bid \$5 for the North and Green bid \$2 for the North. If Yellow claims the North then Yellow will earn \$8 - \$5 = \$3 and Green will earn -\$2. Thus \$3 will be added to Yellow's earnings and Green will have \$2 subtracted from their earnings. However, if Green claims the North then Yellow will earn -\$5 and Green will earn \$8 - \$2 = \$6. Earnings for the other three regions will be determined in the same fashion.

[No Complementarities Phase 3]

Part 3 of this experiment is just like part 2, except that there are now 16 regions on the map and the value of each region is \$3. Regions will be claimed in the same way, your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

[Complementarities Phase 1]

This is an experiment in the economics of decision making. You will be paid in cash at the end of the experiment based upon your decisions, so it is important that you understand the directions completely. Therefore, if you have a question at any point, please raise your hand and someone will assist you. Otherwise we ask that you do not talk or communicate in any other way with anyone else. If you do, you may be asked to leave the experiment and will forfeit any payment.

The experiment will proceed in three parts. You will receive the directions for part 2 after part 1 is completed and for part 3 after part 2 is completed. What happens in part 2 does not depend on what happens in part 1, and what happens in part 3 does not depend on what happens in part 1 or 2. But whatever money you earn in part 1 will carry over to part 2 and whatever you earn in part 2 will carry over to part 3.

Each part contains a series of decision tasks that require you to make choices. For each decision task, you will be randomly matched with another participant in the lab. None of the participants will ever learn the identity of the person they are matched with on any particular round.

In each task you have the opportunity to earn Lab Dollars. At the end of the experiment your lab dollars will be converted into US dollars at the rate \$25 Lab Dollars = \$US 1.

You have been randomly assigned to be either a “Green” or a “Yellow” participant. You will keep this color throughout the experiment. For every task you will be randomly matched with a person who has been assigned the other color. Your color is indicated in a box on the right side of your screen. This portion of your screen also shows you your earnings on a task once it is completed and your cumulative earnings in the experiment. You will start off with an earning balance of \$160.

In the center of your screen you can see a map with four regions: North, South, East, and West. These regions are surrounded by large colored areas. The top left and bottom right areas are colored "Yellow". The top right and bottom left areas are colored "Green". In part 1 of the experiment, when it is your turn you can claim any one unclaimed region on the map by clicking on it. When you claim a region, that part of the map will be colored in with your color. If you can complete a continuous path of regions in your color connecting your two large colored areas you will earn \$48. When the person you are matched with claims a region, that part of the map will be colored in with that person's color. If the person you are matched with completes a continuous path, that person will earn \$48. Exactly one person can complete a path each time and the person that does not complete a path will earn \$0.

On each task Green and Yellow alternate claiming regions until all four regions are claimed. At that point, you will be randomly rematched with another participant for the next task. Which color gets to go first alternates between tasks. You will go through this process several times.

[Complementarities Phase 2]

Part 2 of this experiment is very similar to part 1. The map is the same, as is the value of completing a path. Your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

What is different is how the regions are claimed. Now you and the person you are matched up with have to bid for each of the four regions at the same time. However, any amount you bid on a region is deducted from your earnings regardless of whether or not you get to claim the region. Since you have to pay what you bid, the sum of your bids for the four regions cannot exceed the amount of earnings you have when placing your bids.

Bidding for a region works in the following way. The chance that you claim a region is proportional to how much you bid relative to the total amount bid for that region. For example, suppose that Yellow bid \$6 for the North and Green bid \$2 for the North then the chance that Yellow would claim North is 6/(6+2) = 6/8 = 75%. The chance that Green would claim North is 2/(6+2) = 2/8

= 25%.

As another example, suppose that Yellow bid \$0 for the North and Green bid \$0.25 for the North then the chance that Yellow would claim North is 0/(0+0.25) = 0%. The chance that Green would claim North is 0.25/(0+0.25) = 100%.

If both bidders bid \$0 then each would claim the region with a 50% chance.

You and the person you are matched with will both privately and simultaneously place your bids for all four regions at one time. The computer will then determine who claims each region based upon the probabilities associated with the bids. Each region will turn Yellow or Green to indicate who claimed that region.

As before, whoever completes a path will receive the value for it and have that value added to their earnings. Suppose Yellow bid a total of \$23 on the four regions and Green bid a total of \$18 for the four regions. If Yellow completes a path then Yellow will earn \$48 - \$23 = \$25 and Green will earn -\$18. Thus \$25 will be added to Yellow's earnings and Green will have \$18 subtracted from their earnings. However, if Green completes a path then Yellow will earn -\$23 and Yellow will earn \$48 - \$18 = \$30.

[Complementarities Phase 3]

Part 3 of this experiment is just like part 2, except that there are now 16 regions on the map. Regions will be claimed in the same way, a completed path is still worth \$48, your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

7 Appendix B

Consider a variation on All-Pay Hex in which each player is restricted to bidding on only two cells. As before, the value of the prize is normalized to 1. Given that players now bid on only two cells, it is possible for players to have winning and non-winning sets. For example, bidding on North and South constitutes a winning set for both players. We will call this the "line set" and denote this by L. For player X there are two "diagonal winning sets", consisting either of the cells North and East (denoted D_1^X) or South and West (denoted D_2^X). We will drop the superscript X when it is not needed. Observe that player X also has 3 non-winning sets consisting of the cells East and West, North and West, or South and East..

Similarly for player Y there are two "diagonal winning sets" consisting of either North and West (denoted D_1^Y) or South and East (denoted D_2^Y). Moreover, just as player X, player Y also has 3 non-winning sets. We now make a few observations before proceeding further.

Observation 1: In equilibrium, no player will ever bid on a non-winning set. Take the total bid on such a non-winning set. By putting the same bid on a winning set, a player can increase their expected payoff regardless of the strategy of the other player. Hence bidding on a non-winning set is strictly dominated and cannot be a part of an equilibrium strategy. Recall that the probability of winning is 0 for a non-winning set, while a winning set will have a positive probability of winning.

Observation 2: On a diagonal winning set, there is a symmetry between player X and Y's choices for East and West as well as for North and South.

Observation 3: Each player is indifferent between their two diagonal winning sets. To see this, suppose player Y chooses North and South. Then if X chooses North and East, X has to compete with Y for North and if X chooses South and West, she has to compete with Y for South. Suppose Y chooses North and West. Then if X chooses North and East, she has to compete with Y for North and if she chooses South and West, she has to compete with Y for West. In either case she has to compete with Y for one cell. The same is true when Y chooses South and East. If X favors North and East over South and West, Y can adjust her bids to favor the one cell which is more likely to be the one being competed over. Thus Y will increase her chances of winning, reducing X's chances of winning, and thus X's expected value. Thus our assertion holds.

Proposition: In every equilibrium of the 2-cell game, each player will bid $\frac{1}{8}$ on each of the two cells

Proof: The proof needs to consider two possible cases. Without loss of generality we will first consider the case where player X plays a diagonal set, followed by what happens when X plays a line set. For each case we will allow player Y to play a diagonal or a line set.

Let X_{Di} be the amount player X bids on cell i when she plays a diagonal set and X_{Li} be the corresponding bid when she plays the line set. Player Y's bids for the same sets are denoted by Y_{Di} and Y_{Li} respectively. Let p_{XL} and p_{YL} respectively be the probability that players X and Y play the line set. Given that player X is indifferent between D_1^X and D_2^X, let p_{XD} be the identical probabilities that player X plays either of these diagonal sets. Hence $p_{XL} + 2p_{XD} = 1$. Similarly, $p_{YL} + 2p_{YD} = 1$

Case 1: Diagonal Set

without loss of generality, assume that X is bidding on the set North and East. Suppose Y plays the line set. Thus X will win East with probability p_{YL}, and player Y will win South with the same probability. Moreover, with probability p_{YL} the winner of North will form the winning set. This gives us the first term in equation (1). If player Y choose a diagonal set, then by Observation 3 both North and West or South and East are equiprobable. Again, the cell on which both players put positive bids becomes the key cell. This gives us the second and third terms in equation (1). The last two terms denote the cost of bidding on this set for player X.

$$\Pi_X(D,.) = p_{YL}\left(\frac{X_{DN}}{X_{DN}+Y_{LN}}\right) + p_{YD}\left(\frac{X_{DN}}{X_{DN}+Y_{DN}}\right) + p_{YD}\left(\frac{X_{DE}}{X_{DE}+Y_{DE}}\right) - X_{DE} - X_{DN} \quad (1)$$

Taking the derivatives with respect to X_{DN} and X_{DE} gives us the first order conditions

$$\frac{\partial \Pi_X}{\partial X_{DN}} = p_{YL}\left(\frac{Y_{LN}}{(X_{DN}+Y_{LN})^2}\right) + p_{YD}\left(\frac{Y_{DN}}{(X_{DN}+Y_{DN})^2}\right) = 1 \quad (2)$$

$$\frac{\partial \Pi_X}{\partial X_{DE}} = p_{YD}\left(\frac{Y_{DE}}{(X_{DE}+Y_{DE})^2}\right) = 1 \quad (3)$$

Recall that $X_{DE} = Y_{DE}$ in equilibrium. Hence we can rewrite (3) as $X_{DE} = \frac{1}{8}p_{YD}$

Case 2: North and South

If player X instead chooses to play North and South, each of North and South will have the same probability $\frac{1}{2}p_{YL}$ of being critical. This gives us the first two terms in equation (4). Next, with probability p_{YD}, each of North and South has an equal probability of being critical. This gives us the third and fourth terms in equation (4). Subtracting player X's bids we have

$$\frac{1}{2}p_{YL}\frac{X_{LN}}{X_{LN}+Y_{LN}}+\frac{1}{2}p_{YL}\frac{X_{LS}}{X_{LS}+Y_{LS}}+p_{YD}\frac{X_{LN}}{X_{LN}+Y_{DN}}+p_{YD}\frac{X_{LS}}{X_{LS}+Y_{DS}}-X_{LN}-X_{LS} \quad (4)$$

For the first order conditions we get

$$\frac{\partial \Pi_X}{\partial X_{LN}}=\frac{1}{2}p_{YL}\frac{Y_{LN}}{(X_{LN}+Y_{LN})^2}+p_{YD}\frac{Y_{DN}}{(X_{LN}+Y_{DN})^2}=1 \quad (5)$$

$$\frac{\partial \Pi_X}{\partial X_{LS}}=\frac{1}{2}p_{YL}\frac{Y_{LS}}{(X_{LS}+Y_{LS})^2}+p_{YD}\frac{Y_{DS}}{(X_{LS}+Y_{DS})^2}=1 \quad (6)$$

We thus clearly see $X_{LN}=X_{LS}$. In equilibrium, due to symmetry, $X_{LN}=Y_{LN}$, $X_{DN}=Y_{DN}$, and $X_{DE}=Y_{DE}$. We also know that $2p_{YD}+p_{YL}=1$ as these are the only two options for player Y. This leaves us the following system of equations

$$\begin{aligned}
p_{YL}\left(\tfrac{X_{LN}}{(X_{DN}+X_{LN})^2}\right)+p_{YD}\left(\tfrac{X_{DN}}{(X_{DN}+X_{DN})^2}\right) &= 1\\
\tfrac{1}{2}p_{YL}\tfrac{X_{LN}}{(X_{LN}+X_{LN})^2}+p_{YD}\tfrac{X_{DN}}{(X_{LN}+X_{DN})^2} &= 1\\
X_{DE} &= \tfrac{1}{4}p_{YD}\\
2p_{YD}+p_{YL} &= 1
\end{aligned}$$

Solving this system of equations numerically, we obtain several solutions, however, only two that satisfy having non-negative bids and non-negative probabilities. One of these is $X_{LN}=\frac{1}{8}$, $X_{DN}=\frac{1}{8}$, $X_{DE}=\frac{1}{8}$, and $p_{YD}=\frac{1}{2}$. The other possibility is $X_{LN}=\frac{1}{8}$, $X_{DN}=\frac{2\sqrt{2}-1}{8}$, $X_{DE}=0$, and $p_{YD}=0$. Again, due to symmetry, $p_{YD}=p_{XD}$. Both these cases result in the non-zero bids being $\frac{1}{8}$ the value of the prize in equilibrium.[4] As these are the only critical points with positive expected value, and for bids greater than 1 or of 0 the expected values are non-positive, these must be the only maxima.[5] □

[4] The bids on X_{DN} and X_{DE} in the second solution are irrelevant, as p_{XD}, the probability of playing them, is zero.

[5] This argument is similar to the one on page 23 in Kovenock et al. (2015).